NSTIA Mathematics, course B **Maria Ubiali**
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Mathematical Methods I: Examples II

This is the second of two examples sheets for the Michaelmas term course. As with the first, you are not necessarily expected to attempt all the questions; supervisors will be able to advise about selection of questions. Numerical answers to the *basic skills* questions are given at the end of the sheet while those for the *main questions* will be distributed at the end of term.

Comments and suggestions are welcome (please email to mu227@cam.ac.uk).

P. Integration

Basic skills

P1. Evaluate the following indefinite integrals:

(a) $\int x^2\,dx$ (b) $\int (ax^n + bx + c)\,dx$ (c) $\int e^{2x}\,dx$

(d) $\int (1/x)\,dx$ (e) $\int \sin y\,dy$ (f) $\int x\cos x\,dx$

(g) $\int \sec^2 x\,dx$ (h) $\int (2\cos^2 x - 1)\,dx.$

P2. Calculate the following definite integrals:

(a) $\int_0^3 (x^2+4)\,dx$ (b) $\int_0^2 (x-a)^2\,dx$ (c) $\int_0^\pi e^{i\theta}\,d\theta$

(d) $\int_0^\pi \cos x\,dx$ (e) $\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$ (f) $\int_{-\pi/2}^{\pi/2} x\sin(2x)\,dx\,.$

P3. Express the following in terms of partial fractions:

(a) $\dfrac{1}{1-x^2}$ (b) $\dfrac{3x}{2x^2+x-1}$

(c) $\dfrac{2(1-x^2)}{1+x-x^2-x^3}$ (d) $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}\,.$

Main questions

P4. Evaluate the integral

$$\int_1^2 \frac{3}{2x^2+x-1}\,dx$$

by decomposing the integrand into partial fractions.

P5. Find the indefinite integrals of the following functions:

(a) e^x	(b) $\sinh x$	(c) $\cosh x$	(d) $\tanh x$
(e) $\ln x$	(f) $\cosh^{-1} x$	(g) $\tanh^{-1} x$.	

P6. With the help of suitable substitutions, find the indefinite integrals:

(a) $\displaystyle\int \tan\theta\sqrt{\sec\theta}\, d\theta$ (b) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$ (c) $\displaystyle\int \frac{8-2x}{\sqrt{6x-x^2}}\, dx$

(d) $\displaystyle\int \frac{1}{\sin x}\, dx$ (e) $\displaystyle\int \sec x \tan x\, dx$.

P7. Find the integral

$$\int_a^b \frac{dx}{\sqrt{x^2+x+1}} .$$

(Hint: use a hyperbolic angle substitution.)

P8. By writing both $\sin x$ and $\sinh x$ as exponentials, calculate $\int \sin x \sinh x\, dx$.

P9. Using integration by parts, or otherwise, find the indefinite integrals:

(a) $\displaystyle\int \ln x^3\, dx$ (b) $\displaystyle\int (\ln x)^3\, dx$.

P10. Let

$$I \equiv \int_0^x e^{at} \cos bt\, dt \quad \text{and} \quad J \equiv \int_0^x e^{at} \sin bt\, dt ,$$

where a and b are (real) constants. Using integration by parts, show that

$$I = b^{-1} e^{ax} \sin bx - ab^{-1} J .$$

Find another similar relationship between I and J and hence find I and J separately. Evaluate the (complex) integral $I + iJ$ directly and hence find I and J as its real and imaginary parts, respectively.

P11. By means of a sketch show that $\sin x \geq 2x/\pi$ for $0 \leq x \leq \pi/2$. Hence show that

$$\int_0^{\pi/2} \frac{x^2}{1+\sin^2 x}\, dx < \frac{\pi^3}{8}\left(1 - \frac{\pi}{4}\right) .$$

P12. Consider $F(s) \equiv \int_0^{1+s} f(x;s)\, dx$.

i. Give an expression for dF/ds.

ii. If $f(x;s) = x(s-x)$, determine dF/ds using your expression in (i).

iii. For this $f(x;s)$, calculate the integral $F(s)$ directly and then differentiate with respect to s to confirm your answer to (ii).

iv. For $f(x;s) = x(s-x)$, state the values of s for which $dF/ds = 0$.

P13. By considering the area under the curve $y = \ln x$, show that

$$n \ln n - n < \ln n! < (n+1)\ln(n+1) - n$$

for n a positive integer. Hence show that

$$\ln n! - n \ln n + n < \ln(1 + 1/n)^n + \ln(1+n)\,.$$

P14. A particle is located between x and $x + dx$ with probability $P(x)dx$. If

$$\langle |x|^n \rangle \equiv \int_{-\infty}^{\infty} |x|^n P(x)\, dx\,,$$

show, using Schwarz's inequality, that $\langle |x| \rangle^2 \leq \langle |x|^2 \rangle$. (You might prefer to wait for the lectures on probability before doing this question.)

P15. If

$$I = \int_0^{\pi/2} \frac{\sin x}{\sqrt{x^2+1}}\, dx\,,$$

obtain an upper bound for I using: (i) Schwarz's inequality; and (ii) the inequality $x \geq \sin x$ for $x \geq 0$. Which gives the tighter bound?

Q. Multiple and Gaussian integrals

Q1. Evaluate the integral

$$\int_{x=0}^{2} \int_{y=x/2}^{1} 2xy^2\, dy\, dx\,.$$

Reevaluate the integral changing the order of integration (i.e. integrate first with respect to x then y). You should get the same answer!

Q2. By evaluating appropriate triple integrals find:

i. the volume of a sphere of radius a;

ii. the volume of a flat-topped cone described in cylindrical-polar coordinates by $0 \leq z \leq a$, $0 \leq \phi < 2\pi$ and $0 \leq r \leq z$; and

iii. the volume of a round-topped cone described in spherical-polar coordinates by $0 \leq r \leq a$, $0 \leq \phi < 2\pi$ and $0 \leq \theta \leq \pi/4$.

Q3. Integrate both $\sin(x+y)$ and xe^{xy}:

i. over the square $0 \le x \le \pi/2$ and $0 \le y \le \pi/2$;

ii. over the square $0 \le x \le \pi$ and $0 \le y \le \pi$; and

iii. over the triangle formed by the lines $y = 0$, $x = \pi/2$ and $x = y$.

(Leave your answer in the form of a single integral in the one case in which it cannot be evaluated by standard means.)

Q4. Sketch the curve given in plane-polar coordinates by $r = 2a(1+\cos\phi)$ and evaluate

$$\int\int_D \sqrt{x^2+y^2}\, dx\, dy$$

where D is the interior of the curve.

Q5. Evaluate the integral

$$\int\int x^2(1-x^2-y^2)\, dx\, dy$$

over the interior of the circle of radius 1 centred on the origin.

Q6. Show that

$$\int\int_D x^2 y\, dx\, dy = \frac{32\sqrt{2}-11}{42}$$

where D is the smaller of the two areas bounded by the curves $xy = 1$, $y = x^2$ and $y = 2$.

Q7. By means of mathematical induction show that, for positive n and positive a,

$$\int_0^\infty x^{2n} e^{-ax^2}\, dx = \frac{(2n-1)(2n-3)\dots 1}{2(2a)^n}\sqrt{\frac{\pi}{a}}\,.$$

Verify this result by differentiating $\int_0^\infty e^{-ax^2}\, dx$ with respect to a n times.

Q8. Evaluate the volume integral of $e^{-(x^2+y^2+z^2)/a^2}$ over the whole of three-dimensional space.

Q9. According to the Schrödinger equation of quantum mechanics, an electron in the ground state of a hydrogen atom has a spherically-symmetric spatial probability density distribution $P(r) = Ke^{-2r/a}$, i.e. the probability of locating the electron in a volume dV at radius r is $P(r)dV$. Here, r is the distance of the electron from the proton, a is a fixed constant with dimension of length and K is a (dimensional) constant chosen to make the probability of finding the electron anywhere equal to unity. Calculate:

i. the constant K;

ii. the average value of r; and

iii. the most probable value of r.

R. Probability

R1. Two balls are drawn (without replacement) from a box containing five blue balls, four green and one yellow. Like-coloured balls cannot be distinguished. Describe the sample space of unordered outcomes and calculate the probability of each outcome.

R2. A box of 100 gaskets contains 10 gaskets with type-A defects only, five with type-B defects only and two with both types of defect. Given that a gasket drawn at random has a type-A defect, what is the probability that it also has a type-B defect.

R3. Show that if there are 23 people in a room, the probability that no two of them share the same birthday is less than 50%.

R4. You and a colleague are playing in a gameshow where you both have to select one box each from nine, apparently identical boxes. Six contain a valuable prize but the other three are empty. The host makes you both choose a separate box in turn.

i. If you choose first, what is the probability that you win a prize?

ii. If you choose first and win a prize, what is the probability that your colleague also wins a prize?

iii. If you choose first and do not win a prize, what is the probability that your colleague does win a prize?

iv. Is it in your better interest to persuade your colleague to choose first?

R5. You randomly choose a biscuit from one of two seemingly identical jars. Jar A has 10 chocolate biscuits and 30 plain; jar B has 20 chocolate and 20 plain biscuits. Unfortunately, you choose a plain biscuit. What is the probability that you chose from jar A? (Hint: use Bayes' theorem.)

R6. A weighted die gives a probability p of throwing each of two, three, four or five, probability $2p$ of throwing a six and probability $p/2$ of throwing one.

i. Calculate p.

ii. Calculate $\langle x \rangle$, the expectation value of the number you throw.

iii. In a single throw, what is the probability of obtaining a number higher than $\langle x \rangle$?

iv. Calculate the variance, σ^2, of the number you throw.

R7. In one of the National Lottery games, six balls are drawn at random from 49 balls, numbered from one to 49. You pick six different numbers. What is the probability that your six numbers match those drawn? What is the probability that exactly r of the numbers you choose match those drawn? What is the probability that five numbers of those you choose match those drawn and that your sixth number matches a "bonus ball" drawn from those remaining after the first six balls are drawn?

R8. You arrive home after a long night out and attempt to open the front door with one of the three keys in your pocket. (Assume that exactly one key will open the door and that if the correct key is selected you will succeed in opening the door.) Let the random variable X be the number of tries that you need to make to open the door if you select keys randomly from your pocket but drop any key that fails onto the ground. Let another random variable Y be the number of tries needed if you again select keys at random but immediately put any that fail back into your pocket. Find the probability distribution for X and Y and evaluate their expectation values $\langle X \rangle$ and $\langle Y \rangle$. (Hint: it may be useful to note that $1 + 2x + 3x^2 + \cdots = (1 - x)^{-2}$ for $|x| < 1$.)

R9. An opaque bag contains 10 green counters and 20 red. One counter is selected at random and then replaced: green scores one and red scores zero. Five draws are made.

i. Using a calculator, or otherwise, calculate p_r, the probability of obtaining a score $r \in \{0, 1, 2, 3, 4, 5\}$. Check that the probabilities sum to unity. Write down the mean $\langle r \rangle$ and variance σ^2 of the scores.

ii. Calculate the probability of obtaining scores in the ranges $\langle r \rangle \pm \sigma/2$ and $\langle r \rangle \pm \sigma$.

iii. The Gaussian approximation of the binomial distribution in (i) is

$$P_1(r) \propto \exp[-9(r - 5/3)^2/20] .$$

Sketch $P_1(r)$ and p_r.

iv. Compare your answers in (ii) with those from $P_1(r)$ (treating r as a continuous random variable for the latter). In what sense is $P_1(r)$ a good approximation to p_r?

v. Which of your answers would have been different had you not replaced the counters after each selection?

vi. Which of your answers would have been different had the bag contained only one green counter and two red counters?

R10. The (dimensionless) size, s, of raindrops during a storm has a probability density function given by

$$\begin{aligned} f(s) &= 10ds^2 && (0 \leq s \leq 0.6) \\ f(s) &= 9d(1-s) && (0.6 \leq s \leq 1) \\ f(s) &= 0 && (s \geq 1)\,, \end{aligned}$$

where d is a constant.

i. Find the value of d and sketch the graph of this distribution.

ii. Write down the most likely size of drop.

iii. Find the mean size of drop.

iv. Determine the probability that the size will be: (a) more than 0.8; and (b) between 0.4 and 0.8.

R11. The lifetime, t, of a bulb in a traffic signal is a random variable with density

$$\begin{aligned} f(t) &= 1 && (1 \leq t \leq 2) \\ f(t) &= 0 && \text{otherwise}\,, \end{aligned}$$

where t is measured in years. What is the probability as a function of y that the bulb fails in less than y years? The traffic signal contains three bulbs. Assuming they fail independently, what is the probability as a function of z that none of the bulbs have to be replaced in z years?

R12. A certain disease is known to afflict one in a thousand people. You take a medical test that is said to be 99% accurate (i.e. it gives the correct result in 99% of cases in which it is used). What is the probability that you actually have the disease if the test indicates that you do? Discuss the assumptions implicit in this question and your answer.

R13. Suppose that n distinguishable particles are placed randomly into N boxes (states). A particular configuration of this system is such that there are n_s particles in state s, where $1 \leq s \leq N$. If the ordering of particles in any particular state does not matter, show that the number of ways of realising a particular configuration is

$$W = n! \prod_{s=1}^{N} \frac{1}{n_s!}\,.$$

(The product symbol $\prod$ is defined so that $\prod_{s=1}^{N} a_s \equiv a_1 a_2 \ldots a_N$.)

R14. *The two-daughter problem.* Your tutor has two children and at least one is a girl. What is the probability that your tutor has a son? Would it make a difference if you knew that the younger child was a girl? (You may assume a 50% chance of any one birth being a boy or a girl.)

Answers to basic-skills questions

P1. (a) $x^3/3$; (b) $ax^{n+1}/(n+1) + bx^2/2 + cx$; (c) $e^{2x}/2$; (d) $\ln x$; (e) $-\cos y$; (f) $\cos x + x \sin x$; (g) $\tan x$; (h) $(\sin 2x)/2$. In all cases, a constant of integration should be added.

P2. (a) 21; (b) $(6a^2 - 12a + 8)/3$; (c) $2i$; (d) 0; (e) 2; (f) $\pi/2$;

P3. (a) $1/[2(1+x)] + 1/[2(1-x)]$; (b) $1/(x+1) + 1/(2x-1)$; (c) $2/(x+1)$; (d) $-1 - 1/(x-1) + 1/(x+1) + (2x+3)/(x^2+2x+3)$